MATHEMATICS PAPER IIB

COORDINATE GEOMETRY AND CALCULUS.

TIME : 3hrs **Max. Marks.75**

Note: This question paper consists of three sections A,B and C.

SECTION A

VERY SHORT ANSWER TYPE QUESTIONS. **10X2 =20**

1. Find the values of a, b if $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle. Also Find the radius and centre of the circle.

2. If θ is the angle between the tangents through a point P to the circle S = 0 then $\tan \frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$ where r is the radius of the circle.

3. Show that circles given by the equations $x^2 + y^2 + 6x - 8y + 12 = 0$; $x^2 + y^2 - 4x + 6y + k = 0$intersect each other orthogonally.

4. Find the equation of tangent to the parabola $y^2 = 16x$ inclined at an angle 60° with its axis and also find the point of contact.

5. Find the eccentricity of the ellipse, in standard form, if its length of the latus rectum is equal to half of its major axis.

6.Evaluate $\int \cos x \cos 3x \, dx$ on R.

7.Evaluate $\int \frac{\sin(\mathrm{Tan}^{-1}x)}{1+x^2} dx, x \in R$

8. Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$

9. Find the area under the curve f(x) = cos x in [0,2π].

10. solve $y - x\frac{dy}{dx} = 5\left(y^2 + \frac{dy}{dx}\right)$

SECTION B

SHORT ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING 5 X 4 = 20

11. Find the equation of the tangents to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to $x + y - 8 = 0$

12. The pole of the line $lx + my + n = 0$ $(n \neq 0)$ with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$.

13. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$

14. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the tangent at P meets the X-axis at T then show that (CN)(CT) = a^2 where C is the centre of the ellipse.

15. Find the equations of tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through (–2, 1).

16. Evaluate $\lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n+2} + ... + \sqrt{n+n}}{n\sqrt{n}}$

17.solve $(2x + 2y + 3)\frac{dy}{dx} = x + y + 1$

SECTION C

LONG ANSWER TYPE QUESTIONS.

ANSWER ANY FIVE OF THE FOLLOWING **5 X 7= 35**

18. Find the equation of the circum circle of the triangle formed by the straight lines given in each of the following.

19. Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends a right angle at the origin.

20. (i) If the coordinates of the ends of a focal chord of the parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then prove that $x_1x_2 = a^2$, $y_1y_2 = -4a^2$.

(ii) For a focal chord PQ of the parabola $y^2 = 4ax$, if SO = l and SQ = l' then prove that $\frac{1}{l} + \frac{1}{l'} = \frac{1}{a}$.

21. Evaluate $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

22. Obtain the reduction formula for $I_n = \int \csc^n x \, dx$, n being a positive integer, n ≥ 2 and deduce the value of $\int \operatorname{cosec}^5 x \, dx$.

23. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$

24. solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$

Solutions:

1. Find the values of a, b if $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle. Also Find the radius and centre of the circle.

Sol. the equation of second degree $ax^2 + 2hy + by^2 + 2gx + 2hy + c = 0$ represents a circle if $a = b$, $h = 0$, $g^2 + f^2 - c \geq 0$

$\therefore ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle if b = 0, a = 3

Equation of circle is $3x^2 + 3y^2 - 5x + 2y - 3 = 0$

$x^2 + y^2 - \frac{5}{3}x + \frac{2}{6}y - 1 = 0$

$g = -\frac{5}{6}; f = \frac{2}{6}; c = -1$

$C = (-g, -f) = \left(\frac{5}{6}, \frac{1}{3}\right)$

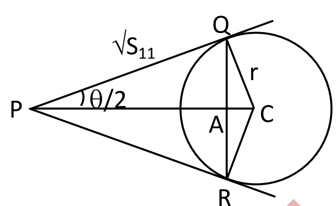
Radius $= \sqrt{g2 + f2 - c}$

$= \sqrt{\frac{25}{36} + \frac{1}{9} + 1} = \frac{\sqrt{65}}{6}$

2. If θ is the angle between the tangents through a point P to the circle S = 0 then $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}}$

where r is the radius of the circle.

Proof :



Let the two tangents from P to the circle S = 0 touch the circle at Q, R and θ be the angle between these two tangents. Let C be the centre of the circle. Now QC = r, PQ = $\sqrt{S_{11}}$ and ΔPQC is a right angled triangle at Q.

$$\therefore \tan\frac{\theta}{2} = \frac{QC}{PQ} = \frac{r}{\sqrt{S_{11}}}$$

3. Show that circles given by the equations $x^2 + y^2 + 6x - 8y + 12 = 0$; $x^2 + y^2 - 4x + 6y + k = 0$intersect each other orthogonally.

Sol. Given circles are

$x^2 + y^2 + 6x - 8y + 12 = 0$; $x^2 + y^2 - 4x + 6y + k = 0$from above circles,

$g = -1$, $f = -1$, $c = -7$, $g^1 = \frac{-4}{3}$, $f^1 = \frac{29}{6}$; $c^1 = 0$. therefore, $c + c^1 = -7 + 0$

$2gg^1 + 2ff^1 = -2(-1)\left(\frac{-4}{3}\right) + 2(-1)\frac{29}{6} = \frac{8}{3} - \frac{29}{3} = \frac{-21}{3} = -7$

Therefore, $2gg^1 + 2ff^1 = c + c^1$

Hence the given circles cut each other orthogonally.

Hence both the circles cut orthogonally.

4. Find the equation of tangent to the parabola $y^2 = 16x$ inclined at an angle 60° with its axis and also find the point of contact.

Sol.

Given parabola $y^2 = 16x$

Inclination of the tangent is

$\theta = 60° \Rightarrow \quad m = \tan 60° = \sqrt{3}$

Therefore equation of the tangent is $y = mx + \frac{a}{m}$

$\Rightarrow y = \sqrt{3}x + \frac{4}{\sqrt{3}}$

$\Rightarrow \sqrt{3}y = 3x + 4$

Point of contact = $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$

5. Find the eccentricity of the ellipse, in standard form, if its length of the latus rectum is equal to half of its major axis.

Sol.

Given, latus rectum is equal to half of its major axis

$\Rightarrow \frac{2b^2}{a} = a$

$2b^2 = a^2$

But $b^2 = a^2(1 - e^2)$

$2a^2(1 - e^2) = a^2$

$1 - e^2 = \frac{1}{2} \Rightarrow e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$

6. $\int \cos x \cos 3x \, dx$ on R.

Sol. $\cos 3x \cos x = \frac{1}{2}(2\cos 3x \cdot \cos x)$

$\frac{1}{2}(\cos 4x + \cos 2x)$

$$\int \cos x \cos 3x\, dx = \frac{1}{2}\int \cos 4x\, dx + \frac{1}{2}\cos 2x\, dx$$

$$= \frac{1}{2}\left(\frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) + C$$

$$= \frac{1}{8}\ \sin 4x + 2\sin 2x\ \ + C$$

7. $\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx, x \in R$

Sol. $\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx$

put $\tan^{-1}x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$$\int \frac{\sin(Tan^{-1}x)}{1+x^2}dx = \int \sin t\, dt$$

$$= -\cos t + t = -\cos(\tan^{-1}x) + C$$

8. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx$ ...(i)

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos(\pi/2 - \pi/2 - x)dx}{1+e^{-x}} \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$ Adding (1) and (2) ,

$$= \int_{-\pi/2}^{\pi/2} \frac{e^x \cos x dx}{1+e^x} ----(2)$$

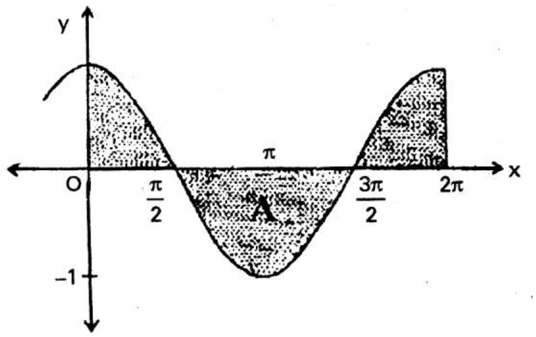
$$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos x(1+e^x)}{1+e^x}dx = \int_{-\pi/2}^{\pi/2} \cos x dx$$

$$2I = 2\int_0^{\pi/2} \cos x\, dx \ \ \because \cos x \text{ is even function}$$

$$\Rightarrow I = \ \sin x\ _0^{\pi/2} \Rightarrow I = 1$$

9. Find the area under the curve f(x) = cos x in [0,2 π].

Sol: We know that $\cos x \ge 0$ in $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \pi\right)$ and ≤ 0 in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$



Required area

$$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(-\cos x\right) dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x dx$$

$$= \left[\sin x\right]_0^{\pi/2} + \left[-\sin x\right]_{\pi/2}^{3\pi/2} + \left[\sin x\right]_{3\pi/2}^{2\pi}$$

$$= \sin\frac{\pi}{2} - \sin 0 - \sin\frac{3\pi}{2} + \sin\frac{\pi}{2} + \sin 2\pi - \sin\frac{3\pi}{2}$$

$$= 1 - 0 - \left(-1\right) + 1 + 0 - \left(-1\right)$$

= 1 + 1 + 1 + 1 = 4.

10. $y - x\frac{dy}{dx} = 5\left(y^2 + \frac{dy}{dx}\right)$

Sol. $y - 5y^2 = (x+5)\frac{dy}{dx} \Rightarrow \frac{dx}{x+5} = \frac{dy}{y(1-5y)}$

Integrating both sides

$$\int \frac{dx}{x+5} = \int \frac{dy}{y(1-5y)} = \int\left(\frac{1}{y} + \frac{5}{1-5y}\right)dy$$

$$\ln|x+5| = \ln y - \ln|1-5y| + \ln c$$

$$\ln|x+5| = \ln\left|\frac{cy}{1-5y}\right| \Rightarrow x+5 = \left(\frac{cy}{1-5y}\right)$$

11. Find the equation of the tangents to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to

$x + y - 8 = 0$

Sol. Equation of the circle is

$S = x^2 + y^2 - 4x + 6y - 12 = 0$

Centre is C(2, -3); r = radius $=\sqrt{4+9+12}=5$
Equation of the given line is x +y-8=0
Equation of the line parallel to above line is x+y+k=0
If x+y+k=0 is a tangent to the circle then
radius = perpendicular distance from the centre.

$5 = \frac{|2-3+k|}{\sqrt{1+1}}$
$\Rightarrow |k-1| = 5\sqrt{2} \Rightarrow k-1 = \pm 5\sqrt{2} \Rightarrow k=1\pm 5\sqrt{2}$
Equation of the tangent is
$x+y+1 \pm 5\sqrt{2} = 0$

12. The pole of the line lx + my + n = 0 (n ≠ 0) with respect to $x^2 + y^2 = a^2$ is $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$.

Proof :

Let $P(x_1, y_1)$ be the pole of lx + my + n = 0 …(1)

The polar of P with respect to the circle is :

$xx_1 + yy_1 - a^2 = 0$ …(2)

Now (1) and (2) represent the same line

$\therefore \frac{x_1}{\ell} = \frac{y_1}{m} = \frac{-a^2}{n} \Rightarrow x_1 = \frac{-la^2}{n}, y = \frac{-ma^2}{n}$

$\therefore$ Pole P = $\left(-\frac{la^2}{n}, -\frac{ma^2}{n}\right)$

13. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle
$2x^2 + 2y^2 + 5x - 6y + 4 = 0$

Sol. Let $S= x^2 + y^2 + 2gx + 2fy + c = 0$
S=0 is passing through (2, 0), (0, 2),
$\Rightarrow$ 4 + 0 + 4g + c = 0 ----(1)
and 0 + 4 + 4f + c = 0 -----(2)
(1) –(2)$\Rightarrow$ f-g =0 $\Rightarrow g = f$
S=0 is orthogonal to $x^2 + y^2 + \frac{5}{2} x - \frac{6}{2} y + 2 = 0$

$\Rightarrow 2g\left(\frac{5}{4}\right) + 2f\left(-\frac{3}{2}\right) = 2 + c$

$\frac{5}{2}g - 3f = 2 + c$
But g = f $\Rightarrow$ $\frac{5}{2} g - 3g = 2 + c$
$\Rightarrow -g = 4 + 2c$
Putting value of g in equation (1)
$-16-8c + c = -4 \Rightarrow$ $c = -\frac{12}{7}$
$\Rightarrow -g\ 4 - \frac{24}{7} = +\frac{4}{7}$

Equation of the circle is

$$x^2 + y^2 - \frac{8x}{7} + \frac{8y}{7} + \frac{12}{7} = 0$$

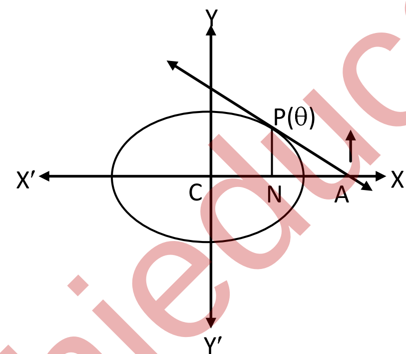
$$\Rightarrow 7(x^2 + y^2) - 8x - 8y - 12 = 0$$

14. If PN is the ordinate of a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the tangent at P meets the X-axis at T then show that (CN)(CT) = a^2 where C is the centre of the ellipse.

Sol: Let P(θ) = (a cos θ, b sin θ) be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then the equation of the tangent at P(θ) is

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ (or)

$\frac{x}{\left(\frac{a}{\cos\theta}\right)} + \frac{y}{\left(\frac{b}{\sin\theta}\right)} = 1$ meets X-axis at T.



∴ X - intercept (CT) = $\frac{a}{\cos\theta}$ and the ordinate of P is PN = b sin θ then its abssicca CN = a cos θ

$$\therefore (CN).(CT) = (a\cos\theta)\frac{a}{\cos\theta} = a^2.$$

15. Find the equations of tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through (–2, 1).

Sol. Equation of the hyperbola is $2x^2 - 3y^2 = 6$

$$\Rightarrow \frac{x^2}{3} - \frac{y^2}{2} = 1$$

Let m be the slope of the tangent .

The tangent id passing through p(–2, 1).

Equation of the tangent is

y – 1 = m(x + 2) = mx + 2m

y = mx + (2m + 1) ...(1)

since (1) is a tangent to the hyperbola,

$c^2 = a^2m^2 - b^2$

$\Rightarrow (2m + 1)^2 = 3m^2 - 2$

$\Rightarrow 4m^2 + 4m + 1 = 3m^2 - 2$

$\Rightarrow m^2 + 4m + 3 = 0 \Rightarrow (m + 1)(m + 3) = 0$

$\Rightarrow m = -1 \text{ or } -3$

Case 1) $m = -1$

Equation of the tangent is

$y = -x-1 \Rightarrow x + y + 1 = 0$

Case 2 $m = -3$

Equation of the tangent is

$y = -3x - 5 \Rightarrow 3x + y + 5 = 0$

16. Evaluate $\lim\limits_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n+2}+...+\sqrt{n+n}}{n\sqrt{n}}$

Sol: For determining the limit we use the result that if f is continuous on [0,1] and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$ is a partition then $\int_0^1 f(x)dx = \lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$

Given $\lim\limits_{n\to\infty}\left(\frac{\sqrt{n+1}+\sqrt{n+2}+...+\sqrt{n+n}}{n\sqrt{n}}\right)$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{\sum_{i=1}^{n}\sqrt{n}\sqrt{\left(1+\frac{1}{n}\right)}+\sqrt{n}\sqrt{\left(1+\frac{2}{n}\right)}+...+\sqrt{n}\sqrt{\left(1+\frac{n}{n}\right)}}{\sqrt{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\sqrt{1+\frac{i}{n}}$$

$$= \int_0^1 \sqrt{1+x}\,dx = \frac{2}{3}\left[(1+x)^{3/2}\right]_0^1$$

$$= \frac{2}{3}\left[2^{3/2}-1\right] = \frac{2}{3}\left[2\sqrt{2}-1\right]$$

17. $(2x + 2y + 3)\frac{dy}{dx} = x + y + 1$

Sol. $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3} = \frac{x+y\ \ +1}{2\ \ x+y\ \ +3}$

Let v = x + y so that $\frac{dv}{dx} = 1 + \frac{dy}{dx}$

$$\frac{dv}{dx} = 1 + \frac{v+1}{2v+3} = \frac{2v+3+v+1}{2v+3} = \frac{3v+4}{2v+3}$$

$$\frac{2v+3}{3v+4} dv = dx$$

$$\frac{2}{3}\int dv + \frac{1}{9}\int \frac{3 \cdot dv}{3v+4} = \int dx$$

$$\frac{2}{3}v + \frac{1}{9}\log(3v+4) = x + c$$

Multiplying with 9

6v + log(3v + 4) = 9x + 9c

6(x + y) + log[3(x + y) + 4] = 9x + c

i.e. log(3x + 3y + 4) = 3x – 6y + c

18. Find the equation of the circum circle of the triangle formed by the straight lines given in each of the following.

i) 2x + y = 4; x + y = 6; x + 2y = 5

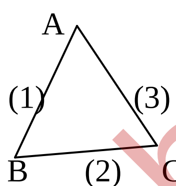
sol.

given lines are

2x + y = 4---------(1)

x + y = 6---------(2)

x + 2y = 5---------(3)



On solving (1) and (2) , we get

B = (-2,8)

On solving (1) and (3) , we get

A = (1,2)

On solving (3) and (2) , we get

C = (7,-1)

Let S(h,k) be the circumcentre of the triangleABC

Then SA = SB = SC.

SA = SB => $SA^2 = SB^2$

=>$(1 – h)^2 + (2 –k)^2 = (-2 – h)^2 + (8 –k)^2$

=> $h^2 + k^2 – 2h-4k+5 = h^2 + k^2 + 4h-16k+68$

=> 6h-12k+63 =0 --------(4)

SA = SC => $SA^2 = SC^2$

=>$(1 – h)^2 + (2 –k)^2 = (7 – h)^2 + (-1 –k)^2$

=> $h^2 + k^2 – 2h-4k+5 = h^2 + k^2-14h+2k+50$

=> 12h-6k -45 = 0--------(5)

Solving (4) and (5), We get S=(17/2,19/2)

Now radius = SA

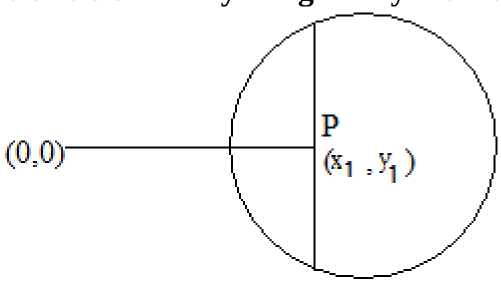
$$= \sqrt{\left(1 - \frac{17}{2}\right)^2 + \left(2 - \frac{19}{2}\right)^2} = \frac{225}{\sqrt{2}}$$

Euation of the circle is

$$\left(x - \frac{17}{2}\right)^2 + \left(y - \frac{19}{2}\right)^2 = \frac{225}{2}$$

$\Rightarrow x^2 + y^2 - 17x - 19y + 50 = 0.$

19.Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle S ≡$x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends a right angle at the origin.



Sol.

Let P (x_1, y_1) be the foot of the perpendicular from the origin on the chord.

Slop a OP = $\frac{y_1}{x_1}$

$\Rightarrow$ Slop of chord = - $\frac{x_1}{y_1}$

$\Rightarrow$ Equation of the chord is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$\Rightarrow yy_1 \quad xx_1 = - xx_1 + x_1^2$

$\Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$

$\Rightarrow \frac{xx_1 + yy_1}{x_1^2 + y_1^2} = 1$ --------(1)

Equation of the circle is $x^2 + y^2 + 2fy + c = 0$ - (2) Hamogenising (2) with the help of (1). Then

$$x^2 + y^2 + (2gx + 2fy)\frac{xx_1 + yy_1}{x_1^2 + y_1^2} + \frac{(xx_1 + yy_1)^2}{(x_1^2 + y_1^2)^2} = 0$$

$$x^2\left[1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)}\right] + y^2\left[1 + \frac{2fy_1}{x_1^2 + y_1^2} + \frac{cy_1^2}{(x_1^2 + y_1^2)}\right] + (\ldots\ldots\ldots\ldots\ldots\ldots)\, xy = 0$$

but above equation is representing a pair of perpendicular lines ,

Co – eff. of x^2 + co-eff of y^2 = 0

$$1 + \frac{2gx_1}{x_1^2 + y_1^2} + \frac{cx_1^2}{(x_1^2 + y_1^2)^2} + 1 + \frac{2fy_1}{x_1^2 + Y_1^2} + \frac{cy_1^2}{(x_1^2 + y_1^2)^2} = 0$$

$$2 + \frac{2gx_1 + 2fy_1}{x_1^2 + y_1^2} + \frac{(x_1^2 + y_1^2)}{(x_1^2 + y_1^2)} = 0$$

$$2 + \frac{2gx_1 + 2fy_1}{x_1^2 + y_1^2} + \frac{c}{x_1^2 + y_1^2} = 0$$

$$2\,(x_1^2 + y_1^2) + 2gx_1 + 2fy_1 + c = 0$$

Locus of L (x_1, y_1) is

$$2(x^2 + y^2) + 2g\,x + 2fy + c = 0$$

20. (i) If the coordinates of the ends of a focal chord of the parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then prove that $x_1x_2 = a^2$, $y_1y_2 = -4a^2$.

(ii) For a focal chord PQ of the parabola $y^2 = 4ax$, if SO = l and SQ = l' then prove that $\frac{1}{l}+\frac{1}{l'}=\frac{1}{a}$.

Sol. i) Let $P(x_1,y_1) = (at_1^2, 2at_1)$ and $Q(x_2, y_2) = (at_2^2, 2at_2)$ be two end points of a focal chord.

P, S, Q are collinear.

Slope of $\overline{PS}$ = Slope of $\overline{QS}$

$$\frac{2at_1}{at_1^2-a}=\frac{2at_2}{at_2^2-a}$$

$$t_1t_2^2-t_1=t_2t_1^2-t_2$$

$$t_1t_2(t_2-t_1)+(t_2-t_1)=0$$

$$1+t_1t_2=0 \Rightarrow t_1t_2=-1$$

From (1)

$$x_1x_2=at_1^2at_2^2=a^2(t_2t_1)^2=a^2$$

$$y_1y_2=2at_1 2at_2=4a^2(t_2t_1)=-4a^2$$

ii) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola, then $t_1t_2 = -1$ (from(1))

$$l=SP=\sqrt{(at_1^2-a)^2+(2at_1-0)^2}$$

$$=a\sqrt{(t_1^2-1)^2+4t_1^2}=a(1+t_1^2)$$

$$l'=SQ=\sqrt{(at_2^2-a)^2+(2at_2-0)^2}$$

$$=a\sqrt{(t_2^2-1)^2+4t_2^2}=a(1+t_2^2)$$

$$\therefore (l-a)(l'-a)=a^2t_1^2t_2^2=a^2(t_1t_2)^2=a^2$$

$$\because t_1t_2=-1$$

$$ll'-a(l+l')=0 \Rightarrow \frac{1}{l}+\frac{1}{l'}=\frac{1}{a}$$

21. $\int \frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$

Sol.

Put cosx = t $\Rightarrow$ –sinx dx = dt

$$\int \frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx=\int \frac{-t\,dt}{t^2+3t+2}$$

$$= -\int \frac{t}{t^2+3t+2}\,dt \qquad ...(1)$$

Let $\frac{t}{t^2+3t+2} = \frac{t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

$\Rightarrow$ t = A(t + 2) + B(t + 1) ...(2)

Put t = −1 in (2)

−1 = A(−1 + 2) $\Rightarrow$ A = −1

Put t = −2 in (2)

−2 = B(−2 + 1) $\Rightarrow$ B = 2

$\therefore \frac{t}{t^2+3t+2} = \frac{-1}{t+1} + \frac{2}{t+2} \qquad ...(3)$

$\therefore$ From (1) and (3)

$$\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx$$

$$= -\left[\int \frac{-1}{t+1}\,dt + 2\int \frac{1}{t+2}\,dt\right]$$

$$= \int \frac{1}{t+1}\,dt - 2\int \frac{1}{t+2}\,dt$$

$$= \log|t+1| - 2\log|t+2| + C$$

$$= \log|1+\cos x| - 2\log|2+\cos x| + C$$

$$= \log|1+\cos x| - \log(2+\cos x)^2 + C$$

$$= \log\left|\frac{1+\cos x}{(2+\cos x)^2}\right| + C$$

22. Obtain the reduction formula for $I_n = \int \csc^n x\,dx$, n being a positive integer, $n \geq 2$ and deduce the value of $\int \operatorname{cosec}^5 x\,dx$.

Sol. $I_n = \int \csc^n x\,dx = \int \csc^{n-2} x \cdot \csc^2 x\,dx$

$$= \csc^{n-2} x(-\cot x) + \int \cot x(n-2)$$
$$\csc^{n-3} x(\cot x)dx$$

$$= -\csc^{n-2} x \cot x + (n-2)\int \csc^{n-2} x$$
$$(\csc^2 x - 1)dx$$

$$= -\csc^{n-2} x \cot x + (n-2)I_{n-2} - (n-2)I_n$$

$$I_n(1+n-2) = -\csc^{n-2} x \cdot \cot x + (n-2)I_{n-2}$$

$$I_n = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$n = 5 \Rightarrow I_5 = -\frac{\csc^3 x \cdot \cot x}{4} + \frac{3}{4} I_3$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2} I_1$$

$$I_1 = \int \csc x \, dx = \log\left|\tan\frac{x}{2}\right|$$

$$I_3 = -\frac{\csc x \cdot \cot x}{2} + \frac{1}{2}\log\left|\tan\frac{x}{2}\right|$$

$$I_5 = -\frac{\csc^3 x \cdot \cot x}{4} - \frac{3}{8}\csc x \cot x + \frac{3}{8}\log\left|\tan\frac{x}{2}\right| + C$$

23. $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$

Sol. $I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$ ----1.

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x}$$ ----2.

Adding 1. and 2.

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

Consider $\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$

Put tan(x/2) = t

$$dx = \frac{2dt}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}$$

$$\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}=\int_0^1\frac{2tdt}{2t+(1-t^2)}$$

$$=2\int_0^1\frac{dt}{(\sqrt{2})^2-(t-1)^2}=2\cdot\frac{1}{2\sqrt{2}}\left[\log\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right]_0^1$$

$$=\frac{1}{\sqrt{2}}\left(\log 1-\log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1)$$

$$I=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

24. $\frac{dy}{dx}(x^2y^3+xy)=1$

Sol. $\frac{dy}{dx}(x^2y^3+xy)=1$

$$\frac{dx}{dy}=xy+x^2y^3$$

$$\Rightarrow\frac{dx}{dy}-xy=x^2y^3 \text{----(1)}$$

Which is Bernoulli's equation

Dividing with x^2,

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{1}{x}y=y^3$$

Put $z=-\frac{1}{x}$ so that $\frac{dz}{dy}=\frac{1}{x^2}\frac{dx}{dy}$

$$\Rightarrow\frac{dz}{dy}+z\cdot y=y^3 \text{ ----2)}$$

which is linear d.eq. in z

$$I.F.=e^{\int ydy}=e^{y^2/2}$$

Sol is $z.I.F=\int Q.\ I.F.\ dy$

$$z\cdot e^{y^2/2}=\int y^3e^{y^2/2}\cdot dy$$

put $\frac{y^2}{2} = t \Rightarrow y\,dy = dt$

$= \int t \cdot dt \cdot e^t = e^t(t-1) = e^{y^2/2}\left(\frac{y^2}{2} - 1\right)$

$z \cdot e^{y^2/2} = e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$

$z = \frac{y^2}{2} - 1 + c \cdot e^{-y^2/2} \Rightarrow -\frac{1}{x} = \frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}$

$-1 = x\left(\frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}\right)$

Hence solution is $1 + x\left(\frac{y^2}{2} - 1 + c \cdot e^{-y^2/2}\right) = 0$